

***-ISOMORPHISM OF LEAVITT PATH ALGEBRAS OVER \mathbb{Z}**

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ABSTRACT. We characterise when the Leavitt path algebras over \mathbb{Z} of two arbitrary countable directed graphs are $*$ -isomorphic by showing that two Leavitt path algebras over \mathbb{Z} are $*$ -isomorphic if and only if the corresponding graph groupoids are isomorphic (if and only if there is a diagonal preserving isomorphism between the corresponding graph C^* -algebras). We also prove that any $*$ -homomorphism between two Leavitt path algebras over \mathbb{Z} maps the diagonal to the diagonal. Both results hold for slight more general subrings of \mathbb{C} than just \mathbb{Z} .

1. INTRODUCTION

Leavitt path algebras were introduced independently in [2] and [5] as algebraic analogues of graph C^* -algebras and have since then attracted a lot of attention, both in connection with graph C^* -algebras and as interesting algebraic objects on their own (they are called Leavitt path algebras because they generalise certain algebras studied by Leavitt in [13, 14, 15]).

The Leavitt path algebra of a directed graph E over a unital commutative ring R is a universal R -algebra $L_R(E)$ whose generators and relations are determined by E (see Section 2.3 for the precise definition of $L_R(E)$). Each involution (for example the identity map) of R gives rise to an involution of $L_R(E)$ which is therefore a $*$ -algebra.

It is natural to ask when two Leavitt path algebras are $(*)$ -isomorphic. Abrams and Tomforde showed in [3] that if two Leavitt path algebras $L_{\mathbb{C}}(E)$ and $L_{\mathbb{C}}(F)$ over \mathbb{C} are $*$ -isomorphic, then so are the corresponding graph C^* -algebras $C^*(E)$ and $C^*(F)$ (their proof is easily generalised to subrings of \mathbb{C} that are closed under complex conjugation and contains 1).

Johansen and Sørensen gave in [11] the first example of two Leavitt path algebras that are not $*$ -isomorphic in spite of the corresponding graph C^* -algebras being isomorphic, when they showed that the Leavitt path algebras over \mathbb{Z} of E_2 and E_{2-} are not $*$ -isomorphic (that $C^*(E_2)$ and $C^*(E_{2-})$ are isomorphic was proved by Rørdam in [17] as an important step towards classifying simple Cuntz-Krieger algebras).

Each Leavitt path algebra $L_R(E)$ contains a certain abelian subalgebra $D_R(E)$ called *the diagonal*. Johansen and Sørensen obtained their result by showing that when E is a finite graph and R is a subring of \mathbb{C} satisfying certain conditions, then every project in $L_R(E)$ belongs to $D_R(E)$. We generalise this result to arbitrary graphs E and slightly more general subrings R of \mathbb{C} (Proposition 5). It follows that any $*$ -homomorphism

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between two Leavitt path algebras $L_{\mathbb{Z}}(E)$ and $L_{\mathbb{Z}}(F)$ over \mathbb{Z} is *diagonal preserving* in the sense that it maps $D_{\mathbb{Z}}(E)$ to $D_{\mathbb{Z}}(F)$ (Corollary 6).

From a countable directed graph E , one can construct a topological groupoid \mathcal{G}_E such that the C^* -algebra of \mathcal{G}_E is isomorphic to the graph C^* -algebra $C^*(E)$, and the Steinberg algebra $A_R(\mathcal{G}_E)$ is isomorphic to $L_R(E)$. It is proved in [7] that two graph C^* -algebras $C^*(E)$ and $C^*(F)$ are isomorphic in a diagonal preserving way if and only if the corresponding graph groupoids \mathcal{G}_E and \mathcal{G}_F are isomorphic, and it is proved in [4] that if every cycle in E and F has an exit, then there is a diagonal preserving isomorphism between $L_R(E)$ and $L_R(F)$ if and only if \mathcal{G}_E and \mathcal{G}_F are isomorphic, and in [6] that if E and F are row-finite graphs with no sinks, then there is a diagonal preserving $*$ -isomorphism between $L_R(E)$ and $L_R(F)$ if and only if \mathcal{G}_E and \mathcal{G}_F are isomorphic.

By using Corollary 6, we show in Theorem 1 that two Leavitt path algebras $L_{\mathbb{Z}}(E)$ and $L_{\mathbb{Z}}(F)$ of countable graphs are $*$ -isomorphic if and only if the groupoids \mathcal{G}_E and \mathcal{G}_F are isomorphic (if and only if there is a diagonal preserving isomorphism between the graph C^* -algebras $C^*(E)$ and $C^*(F)$). As is the case with Proposition 5 and Corollary 6, the result in Theorem 1 holds for slight more general subrings of \mathbb{C} than just \mathbb{Z} .

The rest of the paper is organised in the following way. In Section 2 we recall the definitions of directed graphs, the Leavitt path algebras, graph C^* -algebras, and graph groupoids; and introduce notation. In Section 3 we present our main result (Theorem 1) and discuss how it is related to *orbit equivalence* of graphs and results in [4], [6], and [7] (Remarks 2, 3 and 4). We also ask the question if there exist Leavitt path algebras that are isomorphic without the corresponding graph groupoid being isomorphic. If the answer to this question is “No”, then both *The Isomorphism Conjecture for Graph Algebras* and *The Morita Conjecture for Graph Algebras* introduced in [3] are true. Finally we present and prove in Section 4 Proposition 5 and Corollary 6 before we give the proof of Theorem 1.

2. DEFINITIONS AND NOTATION

We recall in this section the definition of a directed graph, as well as the definitions of the Leavitt path algebra, the graph C^* -algebra, and the graph groupoid of a graph; and introduce some notation. Most of this section is copied from [7].

2.1. Directed graphs. A *directed graph* is a quadruple $E = (E^0, E^1, s, r)$ where E^0 and E^1 are sets, and s and r are maps from E^1 to E^0 . A graph E is said to be *countable* if E^0 and E^1 are countable.

A *path* μ of length n in E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i \leq n-1$. The set of paths of length n is denoted E^n . We denote by $|\mu|$ the length of μ . The range and source maps extend naturally to paths: $s(\mu) := s(\mu_1)$ and $r(\mu) := r(\mu_n)$. We regard the elements of E^0 as path of length 0, and for $v \in E^0$ we set $s(v) := r(v) := v$. For $v \in E^0$ and $n \in \mathbb{N}_0$ we denote by vE^n the set of paths of length n with source v . We define $E^* := \bigcup_{n \in \mathbb{N}_0} E^n$ to be the collection of all paths with finite length. We define $E_{\text{reg}}^0 := \{v \in E^0 : vE^1 \text{ is finite and nonempty}\}$ and $E_{\text{sing}}^0 := E^0 \setminus E_{\text{reg}}^0$. If $\mu = \mu_1 \mu_2 \dots \mu_m$, $v = v_1 v_2 \dots v_n \in E^*$ and $r(\mu) = s(v)$, then we let μv denote the

path $\mu_1\mu_2\cdots\mu_mv_1v_2\cdots v_n$. A *loop* (also called a *cycle*) in E is a path $\mu \in E^*$ such that $|\mu| \geq 1$ and $s(\mu) = r(\mu)$. An edge e is an *exit* to the loop μ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. A graph is said to satisfy *condition (L)* if every loop has an exit.

An *infinite path* in E is an infinite sequence $x_1x_2\cdots$ of edges in E such that $r(e_i) = s(e_{i+1})$ for all i . We let E^∞ be the set of all infinite paths in E . The source map extends to E^∞ in the obvious way. We let $|x| = \infty$ for $x \in E^\infty$. The *boundary path space* of E is the space

$$\partial E := E^\infty \cup \{\mu \in E^* : r(\mu) \in E_{\text{sing}}^0\}.$$

If $\mu = \mu_1\mu_2\cdots\mu_m \in E^*$, $x = x_1x_2\cdots \in E^\infty$ and $r(\mu) = s(x)$, then we let μx denote the infinite path $\mu_1\mu_2\cdots\mu_mx_1x_2\cdots \in E^\infty$.

For $\mu \in E^*$, the *cylinder set* of μ is the set

$$Z(\mu) := \{\mu x \in \partial E : x \in r(\mu)\partial E\},$$

where $r(\mu)\partial E := \{x \in \partial E : r(\mu) = s(x)\}$. Given $\mu \in E^*$ and a finite subset $F \subseteq r(\mu)E^1$ we define

$$Z(\mu \setminus F) := Z(\mu) \setminus \left(\bigcup_{e \in F} Z(\mu e) \right).$$

The boundary path space ∂E is a locally compact Hausdorff space with the topology given by the basis $\{Z(\mu \setminus F) : \mu \in E^*, F \text{ is a finite subset of } r(\mu)E^1\}$, and each such $Z(\mu \setminus F)$ is compact and open (see [19, Theorem 2.1 and Theorem 2.2]).

2.2. Graph C^* -algebras. The *graph C^* -algebra* of a directed graph E is the universal C^* -algebra $C^*(E)$ generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ satisfying

- (CK1) $s_e^*s_e = p_{r(e)}$ for all $e \in E^1$;
- (CK2) $s_es_e^* \leq p_{s(e)}$ for all $e \in E^1$;
- (CK3) $p_v = \sum_{e \in vE^1} s_es_e^*$ for all $v \in E_{\text{reg}}^0$.

If $\mu = \mu_1\cdots\mu_n \in E^n$ and $n \geq 2$, then we let $s_\mu := s_{\mu_1}\cdots s_{\mu_n}$. Likewise, we let $s_v := p_v$ if $v \in E^0$. Then $\text{span}\{s_\mu s_v^* : \mu, v \in E^*, r(\mu) = r(v)\}$ is dense in $C^*(E)$. We define $\mathcal{D}(E)$ to be the closure in $C^*(E)$ of $\text{span}\{s_\mu s_\mu^* : \mu \in E^*\}$. Then $\mathcal{D}(E)$ is an abelian C^* -subalgebra of $C^*(E)$, and it is isomorphic to the C^* -algebra $C_0(\partial E)$. We furthermore have that $\mathcal{D}(E)$ is a maximal abelian sub-algebra of $C^*(E)$ if and only if E satisfies condition (L) (see [16, Example 3.3]).

2.3. Leavitt path algebras. Let E be a directed graph and R a commutative ring with a unit. The *Leavitt path algebra* of E over R is the universal R -algebra $L_R(E)$ generated by pairwise orthogonal idempotents $\{v : v \in E^0\}$ and elements $\{e, e^* : e \in E^1\}$ satisfying

- (LP1) $e^*f = 0$ if $e \neq f$;
- (LP2) $e^*e = r(e)$;
- (LP3) $s(e)e = e = er(e)$;
- (LP4) $e^*s(e) = e^* = r(e)e^*$;

(LP5) $v = \sum_{e \in vE^1} ee^*$ if $v \in E_{\text{reg}}^0$.

If $\mu = \mu_1 \cdots \mu_n \in E^n$ and $n \geq 2$, then we let μ be the element $\mu_1 \cdots \mu_n \in L_R(E)$ and μ^* the element $\mu_n^* \cdots \mu_1^* \in L_R(E)$. For $v \in E^0$, we let $v^* := v$. Then $L_R(E) = \text{span}\{\mu v^* : \mu, v \in E^*, r(\mu) = r(v)\}$. There is a \mathbb{Z} -grading $\bigoplus_{n \in \mathbb{Z}} L_R(E)_n$ of $L_R(E)$ given by $L_R(E)_n = \text{span}\{\mu v^* : \mu, v \in E^*, r(\mu) = r(v), |\mu| - |v| = n\}$ (see [18, Section 3.3]).

We define $D_R(E) := \text{span}\{\mu \mu^* : \mu \in E^*\}$. Then $D_R(E)$ is an abelian subalgebra of $L_R(E)$, and it is maximal abelian if and only if E satisfies condition (L) (see [8, Proposition 3.14 and Theorem 3.22]). If R is a subring of \mathbb{C} that is closed under complex conjugation and contains 1, then $\mu v^* \mapsto v \mu^*$ extends to a conjugate linear involution on $L_R(E)$, i.e. $L_R(E)$ is a $*$ -algebra. There is an injective $*$ -homomorphism $\iota_{L_R(E)} : C^*(E) \rightarrow L_R(E)$ mapping v to p_v and e to s_e for $v \in E^0$ and $e \in E^1$ (see [18, Theorem 7.3]).

2.4. Graph groupoids. Let E be a directed graph. For $n \in \mathbb{N}_0$, let $\partial E^{\geq n} := \{x \in \partial E : |x| \geq n\}$. Then $\partial E^{\geq n} = \bigcup_{\mu \in E^n} Z(\mu)$ is an open subset of ∂E . We define the *shift map* on E to be the map $\sigma_E : \partial E^{\geq 1} \rightarrow \partial E$ given by $\sigma_E(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$ for $x_1 x_2 x_3 \cdots \in \partial E^{\geq 2}$ and $\sigma_E(e) = r(e)$ for $e \in \partial E \cap E^1$. For $n \geq 1$, we let σ_E^n be the n -fold composition of σ_E with itself. We let σ_E^0 denote the identity map on ∂E . Then σ_E^n is a local homeomorphism for all $n \in \mathbb{N}$. When we write $\sigma_E^n(x)$, we implicitly assume that $x \in \partial E^{\geq n}$.

The *graph groupoid* of a countable directed graph is the locally compact, Hausdorff, étale topological groupoid

$$\mathcal{G}_E = \{(x, m-n, y) : x, y \in \partial E, m, n \in \mathbb{N}_0, \text{ and } \sigma^m(x) = \sigma^n(y)\},$$

with product $(x, k, y)(w, l, z) := (x, k+l, z)$ if $y = w$ and undefined otherwise, and inverse given by $(x, k, y)^{-1} := (y, -k, x)$. The topology of \mathcal{G}_E is generated by subsets of the form $Z(U, m, n, V) := \{(x, k, y) \in \mathcal{G}_E : x \in U, k = m-n, y \in V, \sigma_E^m(x) = \sigma_E^n(y)\}$ where $m, n \in \mathbb{N}_0$, U is an open subset of $\partial E^{\geq m}$ such that the restriction of σ_E^m to U is injective, and V is an open subset of $\partial E^{\geq n}$ such that the restriction of σ_E^n to V is injective, and $\sigma_E^m(U) = \sigma_E^n(V)$. The map $x \mapsto (x, 0, x)$ is a homeomorphism from ∂E to the unit space \mathcal{G}_E^0 of \mathcal{G}_E . There is a $*$ -isomorphism from the C^* -algebra of \mathcal{G}_E to $C^*(E)$ that maps $C_0(\mathcal{G}_E^0)$ onto $\mathcal{D}(E)$ (see [7, Proposition 2.2] and [12, Proposition 4.1]), and a $*$ -isomorphism from the Steinberg algebra $A_R(\mathcal{G}_E)$ of \mathcal{G}_E to $L_R(E)$ that maps $\text{span}_R\{1_{Z(\mu), 0, 0, Z(\mu)} : \mu \in E^*\}$ onto $D_R(E)$ (see [6, Theorem 2.2] and [10, Example 3.2]).

3. THE MAIN RESULT

We say that a subring R of \mathbb{C} that is closed under complex conjugation and contains 1 is *kind* if whenever $\lambda_0, \lambda_1, \dots, \lambda_n \in R$ satisfy $\lambda_0 = \sum_{i=0}^n |\lambda_i|^2$, then $\lambda_1 = \dots = \lambda_n = 0$.

Notice that if a subring R of \mathbb{C} is closed under complex conjugation and contains 1 and has an essentially unique partition of the unit as defined in [11], then it is kind (because if $\lambda_0 = \sum_{i=0}^n |\lambda_i|^2$, then $|\lambda_0 - 1|^2 + \sum_{i=1}^n |\lambda_i|^2 + \sum_{i=0}^n |\lambda_i|^2 = 1$). In particular, \mathbb{Z} is kind. The subring of \mathbb{C} generated by 1, π^{-1} and $\sqrt{1 - \pi^{-2}}$ is an example of a kind subring that does not have an essentially unique partition of the unit.

Theorem 1. *Let E and F be countable directed graphs, and let R and S be subrings of \mathbb{C} that are closed under complex conjugation and contain 1, and assume that R is kind. Then the following are equivalent.*

- (1) *The Leavitt path algebras $L_R(E)$ and $L_R(F)$ of E and F are $*$ -isomorphic.*
- (2) *There is a $*$ -isomorphism $\pi : L_S(E) \rightarrow L_S(F)$ such that $\pi(D_S(E)) = D_S(F)$.*
- (3) *There is a $*$ -isomorphism $\phi : C^*(E) \rightarrow C^*(F)$ such that $\phi(\mathcal{D}(E)) = \mathcal{D}(F)$.*
- (4) *The graph groupoids \mathcal{G}_E and \mathcal{G}_F are isomorphic as topological groupoids.*

The proof of Theorem 1 is given in the next section.

Remark 2. It follows from [7] that the following two conditions are equivalent and implied by (3) and (4).

- (5) The pseudogroups \mathcal{P}_E and \mathcal{P}_F introduced in [7, Section 3] are isomorphic.
- (6) E and F are orbit equivalent as in [7, Definition 3.1].

It also follows from [7] that if E and F both satisfy condition (L), then (5) and (6) imply (3) and (4).

Remark 3. Suppose T is a commutative integral domain with a unit. Then (4) implies the following condition.

- (7) There is an isomorphism $\zeta : L_T(E) \rightarrow L_T(F)$ such that $\rho(D_T(E)) = D_T(F)$.

It follows from [4, Corollary 4.4] that if E and F both satisfy condition (L), then (7) implies (3).

Thus, if E and F both satisfy condition (L), then (1)–(7) are all equivalent.

Remark 4. Suppose U is a commutative ring with a unit and an involution $u \mapsto \bar{u}$ that fixes the unit and the zero element. Then there is involution on $L_U(E)$ given by $u\mu v^* \mapsto \bar{u}v\mu^*$ for $u \in U$ and $\mu, v \in E^*$. Thus $L_U(E)$ is a $*$ -algebra.

Condition (4) implies that there is a $*$ -isomorphism $\eta : L_U(E) \rightarrow L_U(F)$ such that $\eta(D_U(E)) = D_U(F)$. If E and F are row-finite with no sinks and U is an integral domain, then [6, Theorem 6.2] shows that the converse holds¹ (see also [6, Corollary 6.5]).

In the light of Theorem 1 and the above remarks, the following question seems natural.

Question. Do there exist directed graphs E and F and a commutative ring R with a unit such that $L_R(E)$ and $L_R(F)$ are isomorphic, but \mathcal{G}_E and \mathcal{G}_F are not isomorphic?

It follows from Theorem 1, [9, Theorem 4.2], and Theorem 5 and part 2 of the remarks following Corollary 7 in [1] that if the answer to the above question is “No”, then both *The Isomorphism Conjecture for Graph Algebras* and *The Morita Conjecture for Graph Algebras* introduced in [3] are true (we cannot rule out the possibility that the conjectures are true even if the above question is “Yes”).

¹[6, Theorem 6.2] says “no sources” rather than “no sinks” because they use the convention that $e^*e = s(e)$ rather than $e^*e = r(e)$

4. PROOF OF THE MAIN RESULT

Let E be a directed graph and R a subring of \mathbb{C} that is closed under complex conjugation and contains 1. As in [11], we say that $p \in L_R(E)$ is a *projection* if $p = p^* = p^2$.

For the proof of Theorem 1 we need the following generalisation of [11, Theorem 5.6].

Proposition 5. *Let E be a directed graph and let R be a subring of \mathbb{C} that is closed under complex conjugation, contains 1 and is kind. If $p \in L_R(E)$ is a projection, then $p \in D_R(E)$.*

Proof. This proof is inspired by the proof of [11, Proposition 4.4].

For $\mu, \nu \in E^*$, we shall write $\mu \leq \nu$ to indicate that there is an $\eta \in E^*$ such that $\mu\eta = \nu$, and $\mu < \nu$ to indicate that $\mu \leq \nu$ and $\mu \neq \nu$.

Since $L_R(E) = \text{span}_{\mathbb{Z}}\{\alpha\beta^* : \alpha, \beta \in E^*\}$, it follows that there are finite subsets A, B of E^* and a family $(\lambda_{(\alpha, \beta)})_{(\alpha, \beta) \in A \times B}$ of elements of R such that

$$p = \sum_{(\alpha, \beta) \in A \times B} \lambda_{(\alpha, \beta)} \alpha\beta^*.$$

By repeatedly replacing $\alpha\beta^*$ by $\sum_{e \in r(\alpha)E^1} \alpha ee^* \beta^*$ if necessary, we can assume that there is a k such that $B \subseteq E^k \cup \{\mu \in E^* : |\mu| < k \text{ and } r(\mu) \in E_{\text{sing}}^0\}$. We can also, by letting some of the $\lambda_{(\alpha, \beta)}$ be 0 if necessary, assume that $B \subseteq A$. Notice that $\alpha\beta^* = 0$ unless $r(\alpha) = r(\beta)$. For $\beta \in B$, let $A_\beta := \{\alpha \in A : r(\alpha) = r(\beta)\}$. We shall also assume that if $\beta \in B$, then there is a least one $\alpha \in A_\beta$ such that $\lambda_{(\alpha, \beta)} \neq 0$ (otherwise we just remove β from B). We claim that $\lambda_{(\alpha, \beta)} = 0$ for all $(\alpha, \beta) \in A \times B$ with $\alpha \in A_\beta \setminus \{\beta\}$, and that $\lambda_{(\alpha, \beta)} = (-1)^{m_\beta}$ for all $\beta \in B$ where m_β is the number of β' 's in B such that $\beta' < \beta$.

Let $B' = \{\beta \in B : \lambda_{(\alpha, \beta)} = 0 \text{ for all } \alpha \in A_\beta \setminus \{\beta\} \text{ and } \lambda_{(\beta, \beta)} = (-1)^{m_\beta}\}$, and suppose $B' \neq B$. Choose $\beta \in B \setminus B'$ such that $\beta' < \beta$ for no $\beta' \in B \setminus B'$. Let

$$F_\beta = \{e \in r(\beta)E^1 : \beta e \leq \beta' \text{ for some } \beta' \in B \setminus \{\beta\}\}$$

and

$$\gamma_\beta = \beta - \beta \sum_{e \in F_\beta} ee^*$$

($F_\beta = \emptyset$ and $\gamma_\beta = \beta$ unless $|\beta| < k$ and $r(\beta)E^1$ is infinite). Then $\beta'^* \gamma_\beta = 0$ for $\beta' \in B$ unless $\beta' \leq \beta$.

Since $p = p^* p$, it follows that

$$(a) \quad \gamma_\beta^* p \gamma_\beta = \gamma_\beta^* p^* p \gamma_\beta.$$

Recall that $L_R(E)$ is \mathbb{Z} -graded. The degree 0 part of the left-hand side of (a) is

$$(b) \quad \sum_{\beta' \in B \leq \beta} \lambda_{(\beta', \beta')} \left(r(\beta) - \sum_{e \in F_\beta} ee^* \right)$$

where $B^{\leq \beta} := \{\beta' \in B : \beta' \leq \beta\}$, and the degree 0 part of the right-hand side of (a) is

$$(c) \quad \left(\left(\sum_{\beta' \in B^{< \beta}} \bar{\lambda}_{(\beta', \beta')} \right) \left(\sum_{\beta' \in B^{< \beta}} \lambda_{(\beta', \beta')} \right) + \sum_{\beta' \in B^{< \beta}} \bar{\lambda}_{(\beta', \beta')} \lambda_{(\beta, \beta)} \right. \\ \left. + \sum_{\beta' \in B^{< \beta}} \lambda_{(\beta', \beta')} \bar{\lambda}_{(\beta, \beta)} + \sum_{\alpha \in A_\beta} |\lambda_{(\alpha, \beta)}|^2 \right) \left(r(\beta) - \sum_{e \in F_\beta} ee^* \right)$$

where $B^{< \beta} := \{\beta' \in B : \beta' < \beta\}$ (we are using here that $\lambda_{(\alpha, \beta')} = 0$ for $\beta' \in B^{< \beta}$ and $\alpha \in A \setminus \{\beta'\}$).

Suppose m_β is even. Then $\sum_{\beta' \in B^{< \beta}} \lambda_{(\beta', \beta')} = 0$ (because $\lambda_{(\beta', \beta')} = (-1)^{m_{\beta'}}$ for each $\beta' \in B^{< \beta}$). Since (b) = (c), it follows that $\lambda_{(\beta, \beta)} = \sum_{\alpha \in A_\beta} |\lambda_{(\alpha, \beta)}|^2$. Since R is kind, it follows that $\lambda_{(\alpha, \beta)} = 0$ for $\alpha \in A_\beta \setminus \{\beta\}$ and $\lambda_{(\beta, \beta)} = 1$ (recall that $\lambda_{(\alpha, \beta)} \neq 0$ for at least one $\alpha \in A_\beta$), but this contradicts the assumption that $\beta \notin B'$.

If m_β is uneven, then $\sum_{\beta' \in B^{< \beta}} \lambda_{(\beta', \beta')} = 1$, so it follows from the equality of (b) and (c) that $1 + \lambda_{(\beta, \beta)} + \bar{\lambda}_{(\beta, \beta)} + \sum_{\alpha \in A_\beta} |\lambda_{(\alpha, \beta)}|^2 = 1 + \lambda_{(\beta, \beta)}$ from which we deduce that $\lambda_{(\alpha, \beta)} = 0$ for $\alpha \in A_\beta \setminus \{\beta\}$ and $\lambda_{(\beta, \beta)} = -1$, and thus that $\beta \in B'$. So we also reach a contradiction in this case.

We conclude that we must have that $B' = B$, and thus that $\lambda_{(\alpha, \beta)} = 0$ for all $(\alpha, \beta) \in A \times B$ with $\alpha \in A_\beta \setminus \{\beta\}$. Since $\alpha\beta^* = 0$ for $\alpha \notin A_\beta$, it follows that $p = \sum_{\beta \in B} \lambda_{(\beta, \beta)} \beta\beta^* \in D_{\mathbb{Z}}(E)$. \square

Corollary 6. *Let E and F be directed graphs, and let R be a subring of \mathbb{C} that is closed under complex conjugation, contains 1 and is kind. If $\pi : L_R(E) \rightarrow L_R(F)$ is a *-homomorphism, then $\pi(D_R(E)) \subseteq D_R(F)$.*

Proof. Follows from Proposition 5 and [11, Proposition 6.1]. \square

Proof of Theorem 1. The equivalence of (3) and (4) is proved in [7], and that (4) implies (1) and (2) follows from [10, Example 3.2].

We shall prove (1) \implies (3) and (2) \implies (3).

(2) \implies (3): We shall closely follow the proof of [11, Lemma 3.5]. Suppose $\pi : L_S(E) \rightarrow L_S(F)$ is a *-isomorphism such that $\pi(D_S(E)) = D_S(F)$. As in the proof of [3, Theorem 4.4], π extends to a *-isomorphism $\phi : C^*(E) \rightarrow C^*(F)$ satisfying $\phi \circ \iota_{L_S(E)} = \iota_{L_S(F)} \circ \pi$. If $\mu \in E^*$, then

$$\phi(s_\mu s_\mu^*) = \phi(\iota_{L_S(E)}(\mu\mu^*)) = \iota_{L_S(F)}(\pi(\mu\mu^*)) \in \iota_{L_S(F)}(D_S(F)) \subseteq \mathcal{D}(F).$$

Since $\mathcal{D}(E)$ is generated by $\{s_\mu s_\mu^* : \mu \in E^*\}$, it follows that $\phi(\mathcal{D}(E)) \subseteq \mathcal{D}(F)$. That $\phi(\mathcal{D}(F)) \subseteq \mathcal{D}(E)$ follows in a similarly way. Thus $\phi(\mathcal{D}(E)) = \mathcal{D}(F)$.

(1) \implies (3): Suppose $\pi : L_R(E) \rightarrow L_R(F)$ is a *-isomorphism. It follows from Corollary 6 that $\pi(D_R(E)) = D_R(F)$, so an application of the implication (2) \implies (3) with $S = R$ shows that (3) holds. \square

REFERENCES

- [1] G. Abrams, P.N. Áhn and L. Márki, *The Leavitt path algebra of a graph*, Rocky Mountain J. Math. **22** (1992), 405–416.
- [2] G. Abrams and G. Aranda-Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2005), 319—334.
- [3] G. Abrams and M. Tomforde, *Isomorphism and Morita equivalence of graph algebras*, Trans. Amer. Math. Soc. **363** (2011), 3733–3767.
- [4] P. Ara, J. Bosa, R. Hazrat and A. Sims, *Reconstruction of graded groupoids from graded Steinberg algebras*, arXiv:1601.02872v1, 19 pages.
- [5] P. Ara, M.A. Moreno and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2007) 157—178.
- [6] J.H. Brown, L. Clark and A. an Huef, *Diagonal-preserving ring *-isomorphisms of Leavitt path algebras*, arXiv:1510.05309v3, 24 pages.
- [7] N. Brownlowe, T.M. Carlsen and M.F. Whittaker, *Graph algebras and orbit equivalence*, to appear in Ergodic Theory Dynam. Systems, doi:10.1017/etds.2015.52, 29 pages.
- [8] C. Gil Canto and A. Nasr-Isfahani, *The maximal commutative subalgebra of a Leavitt path algebra*, arXiv:1510.03992v2, 21 pages.
- [9] T.M. Carlsen, E. Ruiz and A. Sims, *Graph algebras and orbit equivalence*, to appear in Proc. Amer. Math. Soc., doi:10.1090/proc/13321, 11 pages.
- [10] L.O. Clark and A. Sims, *Equivalent groupoids have Morita equivalent Steinberg algebras*, J. Pure Appl. Algebra **219** (2015), 2062—2075.
- [11] R. Johansen, and A.P.W Sørensen, *The Cuntz splice does not preserve *-isomorphism of Leavitt path algebras over \mathbb{Z}* , J. Pure Appl. Algebra **220** (2016), 3966—3983.
- [12] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997) 505—541.
- [13] W.G. Leavitt, *Modules over rings of words*, Proc. Amer. Math. Soc. **7** (1956) 188—193.
- [14] W.G. Leavitt, *Modules without invariant basis number*, Proc. Amer. Math. Soc. **8** (1957) 322—328.
- [15] W.G. Leavitt, *The module type of a ring*, Trans. Amer. Math. Soc. **42** (1962) 113—130.
- [16] G. Nagy and S. Reznikoff, *Pseudo-diagonals and uniqueness theorems*, Proc. Amer. Math. Soc. **142** (2014), 263—275.
- [17] M. Rørdam, *Classification of Cuntz-Krieger algebras*, K-Theory **9** (1995) 31—58.
- [18] M. Tomforde, *Uniqueness theorems and ideal structure for Leavitt path algebras*, J. Algebra **318** (2007) 270—299.
- [19] S. Webster, *The path space of a directed graph*, Proc. Amer. Math. Soc. **142** (2014), 213—225.

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